

## Slower speed and stronger coupling: Adaptive mechanisms of chaos synchronization

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We show that two initially weakly coupled chaotic systems can achieve synchronization by adaptively reducing their speed and/or enhancing the coupling strength. Explicit adaptive algorithms for speed reduction and coupling enhancement are provided. We apply these algorithms to the synchronization of two coupled Lorenz systems. It is found that after a long-time adaptive process, the two coupled chaotic systems can achieve synchronization with almost the minimum required coupling-speed ratio.

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The ability of coupled nonlinear oscillators to synchronize with each other is a basis for the explanation of many processes in nature. Traditionally, synchronization has been understood as the ability of coupled periodic oscillators with different frequencies to switch their behavior from the regime of independent oscillations to the regime of cooperative oscillations, as the strength of the coupling is increased. In recent years, synchronization in coupled chaotic systems has become an important research field with applications in many areas of science and technology [1,2], such as communications [3,4], lasers [5], and chemical and biological systems [6,7]. It has been shown that two coupled identical chaotic systems can change their behavior from uncorrelated chaotic oscillations to identical chaotic oscillations, as the coupling strength is increased [1]. However, this result does not tell us how the coupling strength is increased as two initially weakly coupled systems evolve into synchronization.

Recently, Néda *et al.* proposed a new mechanism for self-adaptive synchronization in coupled periodic oscillators [8]. They investigated the development and dynamics of the rhythmic applause in concert halls. They found that the mechanism lying at the heart of the synchronization process is the period doubling of the clapping rhythm which leads to slower clapping modes.

In this Brief Report, we propose a unified mechanism for adaptive synchronization of initially weakly coupled chaotic systems. We show that two coupled systems can achieve synchronization by adaptively reducing their speed and/or enhancing the coupling strength. We also provide explicit adaptive algorithms for speed reduction and coupling enhancement.

Consider two linearly coupled identical chaotic systems described by

$$\dot{\mathbf{x}}(t) = \tau(t)\mathbf{F}(\mathbf{x}(t)) + \mathbf{K}(t)[\mathbf{y}(t) - \mathbf{x}(t)], \quad (1)$$

$$\dot{\mathbf{y}}(t) = \tau(t)\mathbf{F}(\mathbf{y}(t)) + \mathbf{K}(t)[\mathbf{x}(t) - \mathbf{y}(t)], \quad (2)$$

where  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ ,  $\mathbf{y} = [y_1, y_2, \dots, y_n]^T \in \mathfrak{R}^n$  are the states of the coupled systems.  $\tau(t) > 0$  is called the *speed factor* of the coupled systems. The coupling matrix  $\mathbf{K}(t)$

$= \text{diag}[k_1(t), k_2(t), \dots, k_n(t)]$  is assumed to be a diagonal matrix with  $k_i(t) \equiv k(t) > 0$  for a particular  $i$  and  $k_j(t) \equiv 0$  for  $j \neq i$ . This means that systems (1) and (2) are linearly coupled through their  $i$ th state variables, and  $k(t)$  represents the *coupling strength*. We can rewrite systems (1) and (2) as follows:

$$\dot{\mathbf{x}}(t) = \tau(t) \left\{ \mathbf{F}(\mathbf{x}(t)) + \frac{1}{\tau(t)} \mathbf{K}(t) [\mathbf{y}(t) - \mathbf{x}(t)] \right\}, \quad (3)$$

$$\dot{\mathbf{y}}(t) = \tau(t) \left\{ \mathbf{F}(\mathbf{y}(t)) + \frac{1}{\tau(t)} \mathbf{K}(t) [\mathbf{x}(t) - \mathbf{y}(t)] \right\}. \quad (4)$$

Note that if  $\tau(t) \equiv \tau_0 > 0$  and  $k(t) \equiv k_0 > 0$  are two constants, and the  $(n-1)$ -dimensional subsystem

$$\dot{x}_j(t) = f_j(\mathbf{x}(t)), \quad j = 1, 2, \dots, i-1, i+1, \dots, n \quad (5)$$

is asymptotically stable [2], then there exists a critical value  $\bar{k} > 0$  such that the two coupled systems (3) and (4) will synchronize in the sense that  $\mathbf{x}(t) - \mathbf{y}(t) \rightarrow \mathbf{0}$  as  $t \rightarrow \infty$  when  $k_0/\tau_0 > \bar{k}$ . The dynamics of the synchronization states  $\mathbf{x}(t) = \mathbf{y}(t)$  is governed by

$$\dot{\mathbf{x}}(t) = \tau_0 \mathbf{F}(\mathbf{x}(t)), \quad (6)$$

which has the same attractor for different nonzero values of  $\tau_0$ . However, the value of  $\tau_0$  determines the varying speed of the synchronization state. The lower the value of  $\tau_0$ , the lower the state varying speed. For example, let  $\mathbf{F}$  be the vector of the chaotic Lorenz system [9]. Then Eq. (6) becomes

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \tau_0 \begin{pmatrix} \sigma(x_2 - x_1) \\ rx_1 - x_2 - x_1x_3 \\ x_1x_2 - bx_3 \end{pmatrix}, \quad (7)$$

where we take  $\sigma = 10$ ,  $b = 8/3$ , and  $r = 28$ . For  $\tau_0 = 1, 0.5, 0.2$ , system (7) has the same chaotic attractor but different state varying speed, as shown in Fig. 1.

One may view the coupling-speed ratio  $k/\tau$  as the cost of the coupled systems and the critical value  $\bar{k}$  as the minimum cost required to achieve synchronization. If the value of  $\bar{k}$  is

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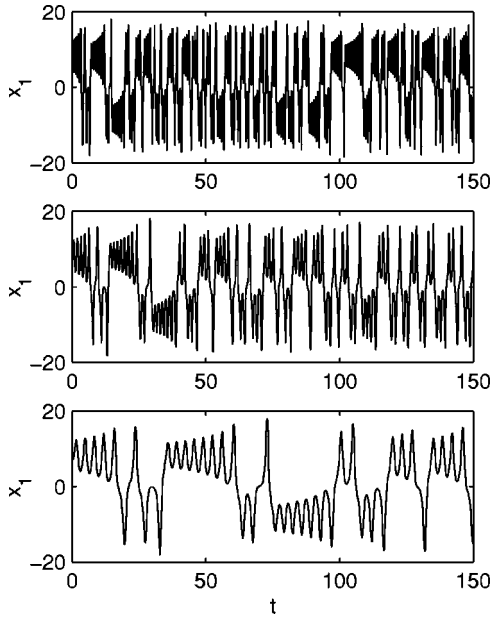


FIG. 1. Trajectories of  $x_1(t)$  of system (7) with  $\tau=1$  (upper), 0.5 (middle), and 0.2 (bottom).

known, then one may take a smaller  $\tau_0$  and/or a larger  $k_0$  so as to guarantee  $k_0/\tau_0 > \bar{k}$ . However, to find the critical value  $\bar{k}$  at the beginning, the complete model of the coupled systems and perhaps the exact initial conditions must be known. Thus it is interesting to see how two initially weakly coupled systems can adaptively update their varying speed and coupling strength according to measured synchronization error until synchronization occurs.

Without loss of generality, we assume  $\tau_0=1$  and  $k_0 < \bar{k}$ . The adaptive algorithms for the update of speed factor and coupling strength are described by

$$\dot{\tau}(t) = \gamma_\tau [x_i(t) - y_i(t)]^2 [\tau^* - \tau(t)], \quad \tau(0) = \tau_0, \quad (8)$$

$$\dot{k}(t) = \gamma_k [x_i(t) - y_i(t)]^2 [k^* - k(t)], \quad k(0) = k_0, \quad (9)$$

where  $\gamma_\tau$  and  $\gamma_k$  are positive adaptive gains.  $\tau^* > 0$  and  $k^* > 0$  represent the minimum allowed speed factor and maximum allowed coupling strength, respectively. Therefore,  $k^*/\tau^*$  represents the maximum allowed coupling-speed ratio (cost) of the coupled systems. A necessary condition for synchronization is  $k^*/\tau^* > \bar{k}$ .

From Eq. (8),  $\tau(t)$  is a decreasing function and  $\tau(t) \geq \tau^*$ ; from Eq. (9),  $k(t)$  is an increasing function and  $k(t) \leq k^*$ . Therefore, there exist two constants  $\hat{\tau}$  and  $\hat{k}$  such that

$$\lim_{t \rightarrow \infty} \tau(t) = \hat{\tau}, \quad \tau^* \leq \hat{\tau} \leq \tau_0, \quad (10)$$

$$\lim_{t \rightarrow \infty} k(t) = \hat{k}, \quad k_0 \leq \hat{k} \leq k^*. \quad (11)$$

If  $\hat{\tau} = \tau^*$  and  $\hat{k} = k^*$ , then  $\hat{k}/\hat{\tau} = k^*/\tau^* > \bar{k}$ , which implies that the coupled systems will achieve asymptotic synchroni-

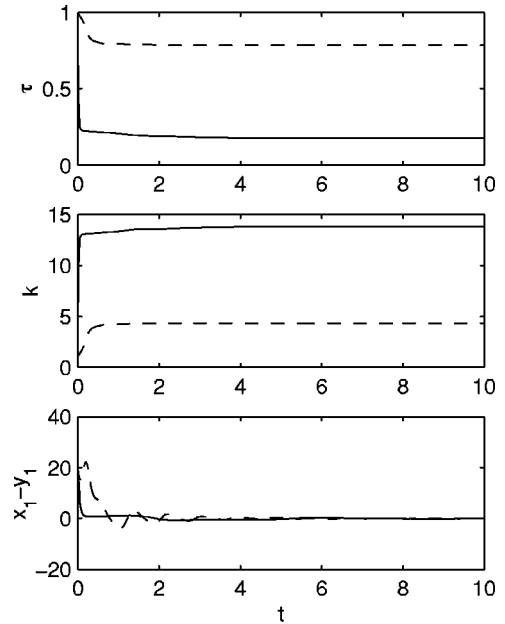


FIG. 2. Adaptive synchronization of the coupled Lorenz systems (13)–(16) with  $\gamma=0.2$  (full line) and 0.002 (dashed line).

zation. On the other hand, if  $\hat{\tau} \neq \tau^*$  and/or  $\hat{k} \neq k^*$ , then from Eqs. (8) and (9) we must have

$$\lim_{t \rightarrow \infty} [x_i(t) - y_i(t)] = 0, \quad (12)$$

which, together with the stability of the subsystem (5), implies that the coupled systems (1) and (2) will also achieve asymptotic synchronization.

As an example, we consider two coupled Lorenz systems described by

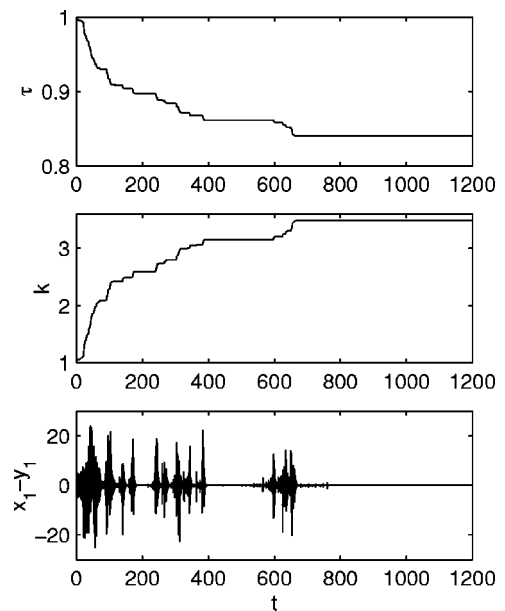


FIG. 3. Adaptive synchronization of the coupled Lorenz systems (13)–(16) with  $\gamma=0.000\ 02$ .

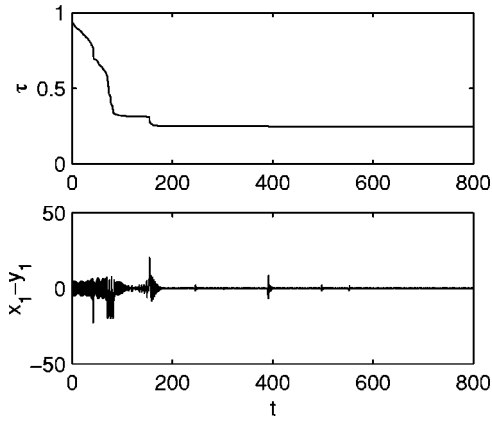


FIG. 4. Synchronization of the coupled Lorenz systems (13) and (14) via adaptive speed-reduction algorithm (15) with gain  $\gamma_\tau = 0.0005$ .

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \tau(t) \begin{pmatrix} \sigma(x_2 - x_1) + k(t)(y_1 - x_1) \\ rx_1 - x_2 - x_1x_3 \\ x_1x_2 - bx_3 \end{pmatrix}, \quad (13)$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \tau(t) \begin{pmatrix} \sigma(y_2 - y_1) + k(t)(x_1 - y_1) \\ ry_1 - y_2 - y_1y_3 \\ y_1y_2 - by_3 \end{pmatrix}, \quad (14)$$

with the following two adaptive algorithms:

$$\dot{\hat{\tau}}(t) = \gamma_\tau (x_1 - y_1)^2 [\tau^* - \tau(t)], \quad (15)$$

$$\dot{\hat{k}}(t) = \gamma_k (x_1 - y_1)^2 [k^* - k(t)]. \quad (16)$$

In the following simulations, the initial states are taken as  $\mathbf{x}(0) = [10, 2, 20]^T$  and  $\mathbf{y}(0) = [-10, -2, 0]^T$ . We find through simulation that the critical value is  $\bar{k} \approx 3.950$ . Other parameters were chosen as follows:  $\tau_0 = 1$ ,  $k_0 = 1$ ,  $\tau^* = 0.1$ , and  $k^* = 15$ .

We take  $\gamma_\tau = \gamma_k = \gamma$ . Figures 2 and 3 show the simulation results for  $\gamma = 0.2$ , 0.002, and 0.00002. For large adaptive gain  $\gamma = 0.2$ ,  $\tau(t)$  decreases fast to  $\hat{\tau} \approx 0.17$  which is close to the minimum allowed value  $\tau^*$  and  $k(t)$  increases fast to  $\hat{k} \approx 13.82$  which is close to the maximum allowed value  $k^*$  (Fig. 2). In this case, although the time to achieve synchronization is quite short (Fig. 2), the cost  $\hat{k}/\hat{\tau} \approx 81.29$  is much higher than the minimum required cost  $\bar{k}$ . On the other hand, for sufficiently small adaptive gain  $\gamma = 0.00002$ ,  $\tau(t)$  decreases to  $\hat{\tau} = 0.84$  and  $k(t)$  increases to  $\hat{k} = 3.48$  very slowly, and therefore the time to achieve synchronization is quite long (Fig. 3); however, in this case, we find that  $\hat{k}/\hat{\tau} \approx 4.14$  which is very close to the critical value  $\bar{k}$ . This implies that after a long-time adaptive process the coupled systems can achieve synchronization with almost minimum required cost. For the middle adaptive gain  $\gamma = 0.002$ ,  $\tau(t)$  decreases fast

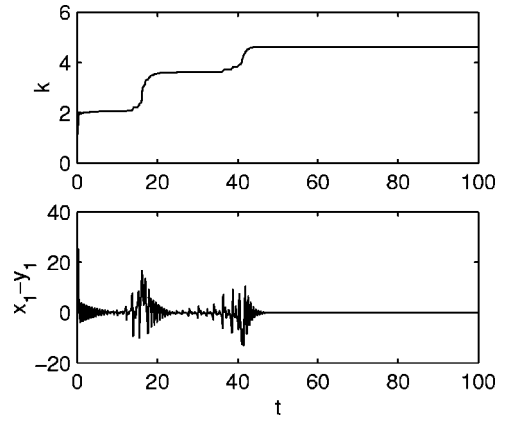


FIG. 5. Synchronization of the coupled Lorenz systems via adaptive coupling-enhancement algorithm (16) with gain  $\gamma_k = 0.0005$ .

to  $\hat{\tau} = 0.78$  and  $k(t)$  increases fast to  $\hat{k} = 4.34$ . In this case, the coupled systems can achieve fast synchronization with low cost  $\hat{k}/\hat{\tau} \approx 5.52$  (Fig. 2).

Now suppose that the coupling strength cannot be changed directly, i.e.,  $k(t) \equiv k_0$ . We show that the coupled systems (13) and (14) can achieve synchronization by adaptively tuning their speed according to algorithm (15) alone. In this case, the critical speed factor is  $\bar{\tau} = k_0/\bar{k} \approx 0.253$ . We find that a large adaptive gain would lead to fast synchronization, but at the same time,  $\hat{\tau}$  is very close to the minimum allowed value  $\tau^*$ , which implies that the varying speed of the synchronization state might be too slow to be efficient. For sufficiently small adaptive gain  $\gamma_\tau$ , the time to achieve synchronization may be quite long, but  $\hat{\tau}$  is very close to the critical value  $\bar{\tau}$ , which implies that after a long-time adaptive process the coupled systems can achieve synchronization and keep the varying speed of the synchronization state as fast as possible. Figure 4 shows the synchronization process for  $\gamma_\tau = 0.0005$ .

If the coupled systems (13) and (14) keep their speed unchanged, i.e.,  $\tau(t) \equiv \tau_0 = 1$ , then they can achieve synchronization by adaptively increasing the coupling strength according to algorithm (16) alone. In this case, the critical coupling strength is  $\bar{k} \approx 3.950$ . A sufficiently large adaptive gain would lead to fast synchronization, but  $\hat{k}$  is very close to the maximum allowed value  $k^*$ , which implies that the coupling strength might be too strong to be safe. For sufficiently small adaptive gain  $\gamma_k$ , the time to achieve synchronization may be quite long, and  $\hat{k}$  is very close to the critical value  $\bar{k}$ , which implies that after a long-time adaptive process the coupled systems can achieve synchronization and keep the coupling strength as small as possible. Figure 5 shows the synchronization process for  $\gamma_k = 0.0005$ .

In summary, we proposed adaptive speed-reduction and/or coupling-enhancement algorithms for synchronization in two initially weakly coupled chaotic systems. We showed that if the adaptive gains are sufficiently small, then after a long-time adaptive process, the two coupled chaotic systems can

achieve synchronization with almost minimum required coupling-speed ratio. Although the speed (time scale) of many physical systems is given *a priori* and is not controllable, it seems to be quite easy for many biological systems to adjust the time scales of their dynamical behaviors. For

example, it is easy for us to change our clapping speeds, and therefore change the time scales of clapping dynamics in concert halls. Therefore, this work might shed some light on the self-adaptive synchronization processes of biological systems.

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